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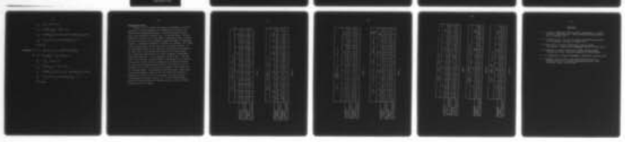

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# ON THE SOLUTION OF CONVEX KNAPSACK PROBLEMS WITH BOUNDED VARIABLES

by

GABRIEL R. BITRAN
AND
ARNALDO C. HAX

Technical Report No. 129

OPERATIONS RESEARCH CENTER





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# ON THE SOLUTION OF CONVEX KNAPSACK PROBLEMS WITH BOUNDED VARIABLES

by


GABRIEL R. BITRAN*

and

ARNOLDO C. HAX


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*Presently at "Universidade de Sao Paulo (Dept. de Engenharia de Producao)", Brazil

## FOREWORD

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Jeremy F. Shapiro
Acting Director

## ABSTRACT

In this paper, ~~we present~~ a recursive method is presented to solve separable differentiable convex knapsack problems with bounded variables. The method differs from classical optimization algorithms of convex programming and determines at each iteration the optimal value of at least one variable. Applications of such problems are frequent in resource allocation and recently have shown to be useful in hierarchical production planning. Computational results are presented.

# ON THE SOLUTION OF CONVEX KNAPSACK PROBLEMS WITH BOUNDED VARIABLES

Gabriel R. Bitran and Arnoldo C. Hax
Massachusetts Institute of Technology

In this paper we present a recursive method to solve the problem

$$(N) \;:\; z = \min \sum_{j\in J^\circ} G_j(x_j)$$

$$\sum_{j\in J^\circ} x_j = P^\circ \qquad (1)$$

$$\ell b_j \leq x_j \leq ub_j \qquad j\in J^\circ \qquad (2)$$

$$x_j \in X_j \qquad j\in J^\circ$$

where $G_j(\cdot)$ $j\in J^\circ$ are differentiable convex functions on the open convex set $X_j \subseteq R$ $j\in J^\circ$ respectively. $ub_j \geq \ell b_j$ $j\in J^\circ$ and $\sum_{j\in J^\circ} \ell b_j < P^\circ < \sum_{j\in J^\circ} ub_j$ (otherwise the problem is trivial or infeasible).

The derivative of $G_j$ at $x_j$ is denoted by $DG_j(x_j)$. The iterative method proposed to solve (N) differs from classical convex programming algorithms and at each iteration determines the optimal value of at least one variable. This problem was formerly treated, for particular objective functions, by convex programming arguments [3] and dynamic programming [6]. More recently Luss and Gupta [4] presented an iterative method for strictly convex decreasing functions and a one pass algorithm for a set of particular functions with the variables bounded from below. Luss and Gupta's method consists in relaxing the upper bounding constraints. The algorithm proposed in this paper applies for a more general set of functions. At each iteration it solves a simpler problem of the form $\min \{\sum G_j(x_j) \text{ st } \sum x_j = P,\ x_j \in X_j\}$ obtained by relaxing both bounds (2). At the end of each iteration (except possibly the last) we show that either the subset of variables $\{x_j : x_j \geq ub_j\}$ have optimal value $ub_j$ in (N) or the subset $\{x_j : x_j \leq \ell b_j\}$ have optimal value $\ell b_j$ in (N). Applications of this class of knapsack problems to resource allocation can be found in [3], [4] and to hierarchical production planning in [2].

(N) is a convex problem and is regular [1]. Thus the Kuhn-Tucker conditions (3)-(9), below, are necessary and sufficient for optimality of $x_j^*$ $j\in J^\circ$ in (N).

$$DG_j(x_j^*) + \lambda + u_j - \tau_j = 0 \qquad j\varepsilon J^\circ \tag{3}$$

$$u_j(x_j^*-ub_j) = 0 \qquad j\varepsilon J^\circ \tag{4}$$

$$\tau_j(\ell b_j-x_j^*) = 0 \qquad j\varepsilon J^\circ \tag{5}$$

$$\sum_{j\varepsilon J^\circ} x_j^* = P^\circ \tag{6}$$

$$\ell b_j \le x_j^* \le ub_j \qquad j\varepsilon J^\circ \tag{7}$$

$$\lambda\varepsilon R,\ u_j \ge 0,\ \tau_j \ge 0 \qquad j\varepsilon J^\circ \tag{8}$$

$$x_j^* \ \varepsilon\ X_j \qquad J\varepsilon J^\circ \tag{9}$$

We first state the algorithm and then prove that it is optimal.

Algorithm

Initialization: $J^1 = J^\circ$, $P^1 = P^\circ$

Iteration $\beta$ ($\beta$=1,2,3...)

Step 1: Solve $N(\beta)$ : $\min \{ \sum_{j\varepsilon J^\beta} G_j(x_j) : \sum_{j\varepsilon J^\beta} x_j = P^\beta,\ x_j\varepsilon X_j\ j\varepsilon J^\beta\}$

and let the solution be $x_j^\beta$ $j\varepsilon J^\beta$. If $\ell b_j \le x_j^\beta \le ub_j$ $j\varepsilon J^\beta$ define $x_j^* = x_j^\beta$ $j\varepsilon J^\beta$ and stop the solution $x_j^*$ $j\varepsilon J^\circ$ generated by the algorithm is optimal. Otherwise go to step 2.

Step 2: Compute

$$\Delta^\beta = \sum_{j\varepsilon J_+^\beta} (x_j^\beta-ub_j) \text{ where } J_+^\beta = \{j\varepsilon J^\beta : x_j^\beta \ge ub_j\}$$

and

$$\nabla^\beta = \sum_{j\varepsilon J_-^\beta} (\ell b_j-x_j^\beta) \text{ where } J_-^\beta = \{j\varepsilon J^\beta : x_j^\beta \le \ell b_j\}$$

Step 3: If $\Delta^\beta \ge \nabla^\beta$ define $x_j^* = ub_j$ $j\varepsilon J_+^\beta$ and let

$$J^{\beta+1} = J^\beta - J_+^\beta,\ P^{\beta+1} = P^\beta - \sum_{j\varepsilon J_+^\beta} ub_j.$$

If $\Delta^\beta < \nabla^\beta$ define $x_j^* = \ell b_j \; j\epsilon J_-^\beta$ and let

$$J^{\beta+1} = J^\beta - J_-^\beta, \; P^{\beta+1} = P^\beta - \sum_{j\epsilon J_-^\beta} \ell b_j .$$

If $J^{\beta+1} = \phi$ stop. The solution $x_j^*$ $j\epsilon J^\circ$ generated by the algorithm is optimal. Otherwise let $\beta = \beta+1$ and go to step 1.

Since at each iteration the set $J^\beta$ is reduced by at least one element the algorithm is finite. To prove that $x_j^*$ $j\epsilon J^\circ$, generated by the algorithm, solve (N) we construct a corresponding Kuhn-Tucker vector through the following results. Let $\lambda^\beta$ be the Kuhn-Tucker multiplier associated to the knapsack contraint in $N(\beta)$.

Lemma 1: If at iteration $\beta$ $\Delta^\beta \geq \nabla^\beta$ then

a) for any $s\epsilon J_+^\beta$ we have $-DG_s(ub_s) \geq -DG_i(ub_i)$ for all $i\epsilon J^\beta - J_+^\beta$

b) $\lambda^{\beta+1} \leq \lambda^\beta$

Proof: a) $\lambda^\beta = -DG_j(x_j^\beta)$ $j\epsilon J^\beta$. Let $s\epsilon J_+^\beta$ and $i\epsilon J^\beta - J_+^\beta$. Then, since for convex functions [5]

$$[DG_j(x_j^2) - DG_j(x_j^1)](x_j^2 - x_j^1) \geq 0 \qquad (10)$$

it follows that

$$-DG_s(ub_s) \geq -DG_s(x_s^\beta) = \lambda^\beta = -DG_i(x_i^\beta) \geq -DG_i(ub_i)$$

b) $$\sum_{j\epsilon J^\beta - J_+^\beta} x_j^\beta = P^\beta - \sum_{j\epsilon J_+^\beta} x_j^\beta \leq P^\beta - \sum_{j\epsilon J_+^\beta} ub_j = P^{\beta+1} = \sum_{j\epsilon J^{\beta+1}} x_j^{\beta+1}$$

For at least one $j\epsilon J^{\beta+1}$ we have $x_j^{\beta+1} \geq x_j^\beta$. Thus

$$\lambda^{\beta+1} = -DG_j(x_j^{\beta+1}) \leq -DG_j(x_j^\beta) = \lambda^\beta$$

where the inequality follows from (10).

Lemma 2: If at iteration $\beta$ $\Delta^\beta < \nabla^\beta$ then

a) for any $s\epsilon J_-^\beta$ we have $-DG_s(\ell b_s) \leq -DG_i(\ell b_i)$ for all $i\epsilon J^\beta - J_-^\beta$

b) $\lambda^{\beta+1} \geq \lambda^\beta$

The proofs of this lemma and of theorem 4 are omitted because they are similar to those of lemma 2 and theorem 3 respectively.

Theorem 3: Assume that $\Delta^{\beta} \geq \nabla^{\beta}$, $\Delta^{i} < \nabla^{i}$ $i=\beta+1,\beta+2,\ldots,\gamma-1$. Then

a) $J^{\gamma} \supseteq (J^{\beta}-J_{+}^{\beta}-J_{-}^{\beta})$

b) $\lambda^{\beta} \geq \lambda^{\gamma}$

Proof:
$$P^{\beta+1} = \sum_{j\epsilon J^{\beta+1}} x_j^{\beta+1} = \sum_{j\epsilon J^{\beta}-J_{+}^{\beta}} x_j^{\beta} - \sum_{j\epsilon J_{+}^{\beta}} ub_j + \sum_{j\epsilon J_{+}^{\beta}} x_j^{\beta}$$

$$= \sum_{j\epsilon J^{\beta+1}} x_j^{\beta} + \Delta^{\beta} \tag{11}$$

$$P^{\beta+2} = \sum_{j\epsilon J^{\beta+2}} x_j^{\beta+2} = \sum_{j\epsilon J^{\beta+1}} x_j^{\beta+1} - \sum_{j\epsilon J_{-}^{\beta+1}} \ell b_j =$$

$$= \sum_{j\epsilon J^{\beta+1}-J_{-}^{\beta+1}} x_j^{\beta} + \sum_{j\epsilon J_{-}^{\beta+1}} x_j^{\beta} + \Delta^{\beta} - \sum_{j\epsilon J_{-}^{\beta+1}} \ell b_j .$$

Thus, since $J^{\beta+2} = J^{\beta+1}-J_{-}^{\beta+1}$

$$P^{\beta+2} = \sum_{j\epsilon J^{\beta+2}} x_j^{\beta} + \Delta^{\beta} - \sum_{j\epsilon J_{-}^{\beta+1}} (\ell b_j - x_j^{\beta}) . \tag{12}$$

Similarly we obtain

$$P^{\gamma} = \sum_{j\epsilon J^{\gamma}} x_j^{\beta} + \Delta^{\beta} - \sum_{s=\beta+1}^{\gamma-1} \sum_{j\epsilon J_{-}^{s}} (\ell b_j - x_j^{\beta}) \tag{13}$$

From (11) it follows that for at least one $j_o \epsilon J^{\beta+1}$

$$x_{j_o}^{\beta+1} \geq x_{j_o}^{\beta} \tag{14}$$

But, from the Kuhn-Tucker conditions for $N(\beta+1)$:

$$-DG_i(x_i^{\beta+1}) = \lambda^{\beta+1} \qquad i\epsilon J^{\beta+1} \tag{15}$$

Combining (10), (14), and (15) it follows that

$$x_j^{\beta+1} \geq x_j^{\beta} \quad \text{for all } j\epsilon J^{\beta+1}$$

If all functions $G_j$ are not strictly convex it is possible that $-DG_{d_o}(x_{d_o}^{\beta+1}) = \lambda^{\beta+1} = \lambda^{\beta}$. In this case $N(\beta+1)$ may have more than one optimal solution. However at least one will satisfy this condition.

Thus $J_-^{\beta+1} \subseteq J_-^{\beta}$ and $\Delta^{\beta} - \sum_{j\epsilon J_-^{\beta+1}} (\ell b_j - x_j^{\beta}) \geq 0.$ (16)

(12), (16) and the fact that $J^{\beta+2} = J^{\beta+1} - J_-^{\beta+1}$ imply that

$$x_j^{\beta+2} \geq x_j^{\beta} \text{ for all } j\epsilon J^{\beta+2},\ J_-^{\beta+2} \subseteq J_-^{\beta} \text{ and}$$

$$\Delta^{\beta} - \sum_{j\epsilon J_-^{\beta+1}} (\ell b_j - x_j^{\beta}) - \sum_{j\epsilon J_-^{\beta+2}} (\ell b_j - x_j^{\beta}) \geq 0 \quad \text{thus}$$

$$P^{\beta+3} = \sum_{j\epsilon J^{\beta+3}} x_j^{\beta+3} \geq \sum_{j\epsilon J^{\beta+3}} x_j^{\beta}$$

Continuing with the same reasoning we obtain

$$J_-^{i} \subseteq J_-^{\beta} \quad i=\beta+1,\dots,\gamma-1;\ \Delta^{\beta} - \sum_{s=\beta+1}^{\gamma-1} \sum_{j\epsilon J_-^{s}} (\ell b_j - x_j^{\beta}) \geq 0$$

and from (13)

$$P^{\gamma} = \sum_{j\epsilon J^{\gamma}} x_j^{\gamma} \geq \sum_{j\epsilon J^{\gamma}} x_j^{\beta}. \qquad (17)$$

These conclusions together with $J^{\beta+1} = J^{\beta} - J_+^{\beta}$ and $J^{i+1} = J^{i} - J_-^{i}$ $i=\beta+1,\dots,\gamma-1$ prove part a).
From (17) it follows $x_j^{\gamma} \geq x_j^{\beta}$ $j\epsilon J^{\gamma}$. Thus, from (10) and (15) with $\gamma$ instead of $\beta+1$

$$\lambda^{\gamma} = -DG_j(x_j^{\gamma}) \leq -DG_j(x_j^{\beta}) = \lambda^{\beta} \qquad ▨$$

<u>Theorem 4</u>: Assume that $\Delta^{\beta} < \nabla^{\beta}$, $\Delta^{i} \geq \nabla^{i}$ $i=\beta+1,\beta+2,\dots,\gamma-1$. Then

a) $J^{\gamma} \supseteq (J^{\beta} - J_-^{\beta} - J_+^{\beta})$

b) $\lambda^{\beta} \leq \lambda^{\gamma}$

<u>Theorem 5</u>: The set $x_j^*$ $j\varepsilon J°$ generated by the algorithm is optimal in (N).

Proof: By lemmas 1 and 2, theorems 3b) and 4b) the set $x_j^*$ $j\varepsilon J°$ generated by the algorithm has the following property:

$$\underbrace{-DG_{k_1}(ub_{k_1}) \geq \ldots \geq -DG_{k_p}(ub_{k_p})}_{\substack{\text{corresponding to variables}\\ \text{defined at upper bound}}} \geq \underbrace{-DG_{v_1}(x^*_{v_1}) = \ldots = -DG_{v_g}(x^*_{v_g})}_{\substack{\text{corresponding to variables}\\ \text{not defined at any iteration}\\ \text{at upper or lower bound}}} \geq \underbrace{-DG_{i_1}(\ell b_{i_1}) \geq \ldots \geq -DG_{i_s}(\ell b_{is})}_{\substack{\text{corresponding to variables}\\ \text{defined at lower bound}}}$$

The $x^*_{v_j}$ $j=1,2,\ldots,g$ are obtained at the last iteration.

To see that conditions (3)-(9) are satisfied take

$$\lambda = -DG_{v_1}(x^*_{v_1})$$

$$\tau_{i_j} = \lambda + DG_{i_j}(\ell b_{i_j}) \geq 0 \quad , \quad u_{i_j} = 0 \qquad j=1,2,\ldots,s$$

$$u_{k_j} = -DG_{k_j}(ub_{k_j}) - \lambda \geq 0 \quad , \quad \tau_{k_j} = 0 \qquad j=1,2,\ldots,p \quad \text{and}$$

$$\tau_{v_j} = u_{v_j} = 0 \qquad j=1,2,\ldots,g.$$

Thus, since $x_j^*$ $j\varepsilon J°$ also satisfies (6), (7), and (9) it follows that

$x_{k_j} = ub_{k_j}$ $j=1,2,\ldots,p$; $x_{v_j} = x^*_{v_j}$ $j=1,2,\ldots,g$ and

$x_{i_j} = \ell b_{i_j}$ $j=1,2,\ldots,s$ solve (N).

The algorithm depends strongly on the existence of a solution to $N(\beta)$. However when among the $G_j(\cdot)$ $j\varepsilon J^\beta$ there are strictly increasing and decreasing functions $N(\beta)$ has no solution. The next theorem shows how to cope with this difficulty.

Let $J_1 = \{j\varepsilon J^o : G_j(\cdot) \text{ is strictly increasing}\} \neq \emptyset$ and

$J_2 = \{j\varepsilon J^o : G_j(\cdot) \text{ is strictly decreasing}\} \neq \emptyset$

Assume that (N) has an optimal solution $x_j^*$ $j\varepsilon J^o$. Then

Theorem 6: The optimal solution of (N) is such that either

$$x_i^* = ub_i \quad i\varepsilon J_2 \text{ and/or } x_i^* = \ell b_i \quad i\varepsilon J_1.$$

Proof: The optimal solution satisfies (3)-(9) and in particular

$$-DG_j(ub_j) = \lambda + u_j \quad j\varepsilon J(ub) \equiv \{j\varepsilon J^o : x_j^* = ub_j\}$$

$$-DG_j(\ell b_j) = \lambda - \tau_j \quad j\varepsilon J(\ell b) = \{j\varepsilon J^o : x_j^* = \ell b_j\} \text{ and}$$

$$-DG_j(x_j^*) = \lambda \quad j\varepsilon J^o - J(\ell b) - J(ub).$$

Note that the assumption $P° > \sum_{j\varepsilon J°} \ell b_j$ implies $J°-J(\ell b) \neq \emptyset$. If $\min\{-DG_j(x_j^*)\ j\varepsilon J°-J(\ell b)\} \geq 0$, since $-DG_i(x_i) < 0 \quad i\varepsilon J_1$, it follows that $J_1 \subseteq J(\ell b)$. Otherwise, since $-DG_i(x_i) > 0 \quad i\varepsilon J_2$ and $-DG_j(ub_j) \geq -DG_i(x_i^*) \geq -DG_k(\ell b_k)$ for any $j\varepsilon J(ub)$ $i\varepsilon J°-J(ub)-J(\ell b)$ and $k\varepsilon J(\ell b)$, we have that $J_2 \subseteq J(ub)$. ▨

A direct consequence of theorem 6 is that the solution to (N) can be obtained by solving the following two problems

$$(N_1) : z_1 = \min\{\sum_{j\varepsilon J°-J_2} G_j(x_j) : \sum_{j\varepsilon J°-J_2} x_j = P° - \sum_{j\varepsilon J_2} ub_j \quad \ell b_j \leq x_j \leq ub_j \quad j\varepsilon J°-J_2\}$$

and

$$(N_2) : z_2 = \min\{\sum_{j\varepsilon J°-J_1} G_j(x_j) : \sum_{j\varepsilon J°-J_1} x_j = P° - \sum_{j\varepsilon J_1} \ell b_j \quad \ell b_j \leq x_j \leq ub_j \quad j\varepsilon J°-J_1\}$$

and taking $z = \min(z_1, z_2)$.

Let $(N\leq)$ and $(N\geq)$ be the versions of (N) when constraint (1) is an inequality of the type $\leq$ and $\geq$ respectively. For a variable $j\varepsilon J^o$ let $h_j$ be the value of $x_j$ for which $DG_j(x_j) = 0$. If such a point does not exist we adopt $h_j = -\infty$ $(+\infty)$ in Theorem 7 (Theorem 8). Let $x_j^*$ $j\varepsilon J^o$ solve (N). Then

Theorem 7: a) If $\lambda = -DG_{v_j}(x_{v_j}^*) \leq 0$, $x_j = x_j^*$ $j\varepsilon J^o$ solve $(N\geq)$.

b) If $\lambda = -DG_{v_j}(x_{v_j}^*) < 0$, $x_j$ defined as:

$x_{i_j} = \ell b_{i_j} \quad j=1,2,\ldots,s;$

$x_{v_j} = \max(\ell b_{v_j}, h_{v_j}) \quad j=1,2,\ldots,g;$

$x_{k_j} = \max(\ell b_{k_j}, h_{k_j})$ for all $k_j$ such that $DG_{k_j}(ub_{k_j}) \geq 0$ and

$x_{k_j} = ub_{k_j}$ for all $k_j$ such that $DT_{k_j}(ub_{k_j}) < 0$

solve ($N\leq$).

<u>Theorem 8</u>: a) If $\lambda = -DG_{v_j}(x^*_{v_j}) \leq 0$, $x_j = x^*_j$ $j \epsilon J^o$ solves ($N\geq$).

b) If $\lambda = -DG_{v_j}(x^*_{v_j}) > 0$, $x_j$ defined as:

$x_{k_j} = ub_{k_j} \quad j=1,2,\ldots,p;$

$x_{v_j} = \min(ub_{v_j}, h_{v_j}) \quad j=1,2,\ldots,g;$

$x_{i_j} = \min(ub_{i_j}, h_{i_j})$ for all $i_j$ such that $DG_{i_j}(\ell b_{i_j}) \leq 0$ and

$x_{i_j} = \ell b_{i_j}$ for all $i_j$ such that $DG_{i_j}(\ell b_{i_j}) > 0$

solve ($N\geq$).

Computational Results

In Tables 1 to 7 we present the results of 84 problems of type (N). The data were randomly generated. For identification purposes, our algorithm was denominated Bitran-Hax. In each problem the objective function was composed by functions $G_i(x_i)$ of the same family. They are indicated in the first row of each table. Following the time in seconds to solve each problem, by both methods, appears the number of iterations required. n represents the number of variables in a problem. For a fixed n three problems were solved for each type of objective function. In Luss and Gupta's method [4] the ordering of the derivatives evaluated at the lower bound of each variable was executed by the "Quick Method". In our algorithm, problems $N(\beta)$ were solved through the corresponding Kuhn-Tucker conditions. For the problems of Tables 6 and 7, Luss and Gupta's algorithm does not apply because in the former one the objective functions are strictly convex increasing and in the later one we have not imposed any condition among the values of the bounds, $P^\circ$ and the point where each of the $G_i(x_i)$ attains its minimum. The computer used is a Borroughs B6700. The programs were written in Algol. Application of problems presented in Tables 5 and 7 to hierarchical production planning can be found in [2]. The parameters ($s_i, m_i, \ell b_i, ub_i$, etc....) corresponding to problems in Tables 1 to 5 (6 and 7) were randomly generated in intervals where the functions $G_i(x_i)$ are strictly convex decreasing (strictly convex).

| | $G_i(x_i) = s_i[1-\exp(-m_i x_i)] \quad i=1,2,\ldots,n$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | n = 50 | | | n = 100 | | | n = 150 | | | n = 200 | | |
| Bitran-Hax (sec.) | 0.138 | 0.157 | 0.119 | 0.292 | 0.213 | 0.237 | 0.482 | 0.540 | 0.434 | 0.709 | 0.720 | 0.711 |
| No. of Iterations | 6 | 7 | 7 | 6 | 6 | 6 | 8 | 9 | 7 | 8 | 8 | 9 |
| Luss-Gupta (sec.) | 0.728 | 1.029 | 1.174 | 1.125 | 4.179 | 0.830 | 5.903 | 9.073 | 3.969 | 10.531 | 2.109 | 15.124 |
| No. of Iterations | 2 | 4 | 8 | 1 | 7 | 1 | 3 | 4 | 2 | 3 | 1 | 5 |

Table 1

| | $G_i(x_i) = s_i \ell n[1+m_i x_i] \quad i=1,2,\ldots,n$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | n = 50 | | | n = 100 | | | n = 150 | | | n = 200 | | |
| Bitran-Hax (sec.) | 0.051 | 0.050 | 0.038 | 0.118 | 0.099 | 0.131 | 0.163 | 0.178 | 0.162 | 0.191 | 0.245 | 0.208 |
| No. of Iterations | 4 | 4 | 3 | 6 | 5 | 6 | 6 | 6 | 4 | 7 | 6 | 5 |
| Luss-Gupta (sec.) | 0.486 | 0.447 | 0.201 | 0.516 | 1.316 | 0.488 | 0.536 | 0.958 | 1.232 | 3.394 | 4.210 | 1.105 |
| No. of Iterations | 3 | 3 | 1 | 2 | 4 | 2 | 1 | 2 | 1 | 6 | 4 | 1 |

Table 2

$$G_i(x_i) = s_i \frac{x_i + c_i}{x_i + m_i} \text{ with } (m_i > c_i) \quad i=1,2,\ldots,n$$

| | n = 50 | | | n = 100 | | | n = 150 | | | n = 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Bitran-Hax (sec.) | 0.113 | 0.095 | 0.124 | 0.208 | 0.221 | 0.178 | 0.308 | 0.307 | 0.298 | 0.400 | 0.421 | 0.327 |
| No. of iterations | 6 | 5 | 5 | 5 | 5 | 5 | 4 | 6 | 4 | 5 | 6 | 3 |
| Luss-Gupta (sec.) | 0.210 | 0.284 | 0.229 | 0.825 | 0.796 | 0.291 | 1.033 | 2.687 | 1.098 | 1.394 | 5.448 | 2.543 |
| No. of iterations | 1 | 1 | 1 | 3 | 2 | 1 | 1 | 4 | 1 | 1 | 2 | 1 |

Table 3

$$G_i(x_i) = x_i x_i - m_i x_i^2 \text{ with } \ell b_i < k_i \text{ and } P < \sum_{i=1}^{n} \mathrm{mn}(ub_i, k_i) \text{ where } k_i = \frac{s_i}{2m_i}$$

| | n = 50 | | | n = 100 | | | n = 150 | | | n = 200 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Bitran-Hax (sec.) | 0.063 | 0.059 | 0.049 | 0.105 | 0.108 | 0.101 | 0.152 | 0.161 | 0.157 | 0.223 | 0.261 | 0.210 |
| No. of iterations | 7 | 7 | 6 | 8 | 7 | 6 | 8 | 8 | 8 | 8 | 8 | 8 |
| Luss-Gupta (sec.) | 0.225 | 0.236 | 0.164 | 0.361 | 0.850 | 0.283 | 0.737 | 1.129 | 0.693 | 2.150 | 0.893 | 0.745 |
| No. of iterations | 16 | 12 | 1 | 5 | 34 | 1 | 10 | 13 | 10 | 18 | 1 | 7 |

Table 4

| | $G_i(x_i) = a_i/x_i \quad i=1,2,\ldots,n$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | n = 50 | | | n = 100 | | | n = 150 | | | n = 200 | | |
| Bitran-Hax (sec.) | 0.061 | 0.058 | 0.048 | 0.095 | 0.076 | 0.133 | 0.128 | 0.246 | 0.168 | 0.276 | 0.306 | 0.282 |
| No. of iterations | 6 | 3 | 4 | 5 | 6 | 7 | 3 | 6 | 5 | 8 | 4 | 3 |
| Luss-Gupta (sec.) | 0.624 | 0.752 | 0.916 | 2.611 | 2.012 | 1.431 | 2.618 | 2.715 | 4.869 | 3.719 | 3.901 | 5.088 |
| No. of iterations | 2 | 1 | 3 | 4 | 2 | 1 | 2 | 5 | 5 | 6 | 2 | 1 |

Table 5

| | $G_i(x_i) = \exp k_i x_i \quad k_i > 0 \quad i=1,2,\ldots,n$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | n = 50 | | | n = 100 | | | n = 150 | | | n = 200 | | |
| Bitran-Hax (sec.) | 0.163 | 0.151 | 0.135 | 0.237 | 0.267 | 0.261 | 0.445 | 0.443 | 0.381 | 0.596 | 0.486 | 0.536 |
| No. of iterations | 7 | 7 | 6 | 5 | 6 | 6 | 7 | 7 | 7 | 8 | 6 | 6 |

Table 6

| | $G_i(x_i) = \frac{1}{2}(\frac{h}{s} - \frac{x_i}{s_i})^2 \quad i=1,2,\ldots,n$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | n = 50 | | | n = 100 | | | n = 150 | | | n = 200 | | |
| Bitran-Hax (sec.) | 0.054 | 0.050 | 0.044 | 0.114 | 0.073 | 0.073 | 0.159 | 0.151 | 0.170 | 0.202 | 0.214 | 0.227 |
| No. of iterations | 6 | 6 | 5 | 6 | 3 | 3 | 5 | 4 | 5 | 4 | 5 | 8 |

Table 7

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